MATH2050C Selected Solution to Assignment 13

Section 5.6 no. 3, 4, 14, 15.

(3) It is clear that both functions are strictly increasing everywhere. Their product $h(x) = x(x-1)$ satisfies $h(0) = h(1) = 0$ so it cannot be increasing on $[0, 1]$. Indeed, if h is increasing, it implies that h is the constant zero function which is clearly ridiculous. In general, the product of two non-negative, increasing functions is increasing.

(4) Let f and g be two positive, increasing function and let $x < y$ be two points in their domain of definition. Then,

$$(fg)(x) - (fg)(y) = f(x)g(x) - f(y)g(y) = (f(x) - f(y))g(x) + f(y)(g(x) - g(y)) \leq 0 ,$$

so fg is increasing.

(14) We claim when $mq = np$,

$$(x^{1/n})^m = (x^{1/q})^p , \quad x > 0, m, p \in \mathbb{Z},\ n, q \in \mathbb{N} .$$

We raise the left hand side by n-th power to get

$$((x^{1/n})^m)^n = (x^{1/n})^{mn} = (x^{1/n})^{nm} = ((x^{1/n})^n)^m = x^m .$$

On the other hand, the n-th power of the right hand side is

$$((x^{1/q})^p)^n = (x^{1/q})^{pn} = (x^{1/q})^{qm} = ((x^{1/q})^q)^m = x^m .$$

So the n-th power of both sides are equal, hence both sides are equal.

Note. So far we define the m/n-th power of x to be $(x^{1/n})^m$. After this problem, it makes sense to define any rational power x^r. So the definition of x^r is independent of the choice of numerator/denumerator in r, that is, x^{m_1/n_1} and x^{m_2/n_2} represent the same number as long as $m_1 n_2 = m_2 n_1$.

(15) First, we claim

$$x^r x^s = x^{r+s} , \quad x > 0,\ r, s \in \mathbb{Q} .$$

Let $r = m_1/n_1$ and $s = n_2/m_2$. Raising to $n_1 n_2$-th power, the right hand side becomes

$$(x^{r+s})^{n_1 n_2} = x^{m_1 n_2 + m_2 n_1} .$$

And the left hand side becomes

$$(x^r x^s)^{n_1 n_2} = (x^r)^{n_1 n_2} (x^s)^{n_1 n_2} = ((x^{m_1/n_1})^{n_1})^{n_2} ((x^{m_2/n_2})^{n_2})^{n_1} = x^{m_1 n_2} x^{m_2 n_1} = x^{m_1 n_2 + m_2 n_1},$$

which is equal to the right hand side.

Next, $(x^r)^s = x^{rs}$. The right hand side is $x^{m_1 m_2/n_1 n_2}$ and so its $n_1 n_2$-th power is $x^{m_1 m_2}$. The $n_1 n_2$-power of the left hand side is

$$((x^{m_1/m_2})^{m_2/n_2})^{n_1 n_2} = (x^{m_1/n_1})^{m_2 n_1} = x^{m_1 m_2} ,$$

so the right and left hand sides are the same.